Probabilities and the St. Petersburg Paradox

Gregory Johnson*

November 17, 2018

Abstract

Two solutions to the St. Petersburg Paradox are described. First, when offered the chance to participate, players may be selecting a *terminal point*: the round in the game that has the lowest probability that a player believes or guesses might occur. Once that is set, the probabilities for the rounds after the player's terminal point are treated as 0. Alternatively, players can discount the probabilities for all possible rounds with the function $prob' = (prob)^m$, $m > 1$. After either modification to the player's evaluation of the probabilities, players may be following the principle of maximizing expected value when they offer only a small payment to enter the game.

1 Introduction

Consider the following game, first described by Nicolaus Bernoulli and then published by his brother Daniel (1738). To participate, a player has to make a payment before the game begins. Then, a fair coin will be flipped until it lands showing heads. If the first flip is heads, then the player gets \$2 and the game ends. If it is tails, the player gets nothing and the game continues. If the coin is heads on the second flip, the player gets \$4 and the game ends. If it is tails, the player gets nothing and the game continues. And so on with the payoff doubling each round. This is the St. Petersburg game, and the question is how much should a player pay to enter the game?

*gregory.johnson@msstate.edu

The expected value of the game is

$$\sum_{k=1}^{n} \left(\frac{1}{2}\right)^{k} \left(2^{k}\right) = \frac{1}{2}(2) + \frac{1}{4}(4) + \frac{1}{8}(8) + \ldots = 1 + 1 + 1 + \ldots = \infty. \quad (1)$$

Since the expected value is infinite, a rational player should be willing to pay any finite amount to play. Clearly, though, no one would pay a very large amount of money, and most people would be willing to pay very little. Hence, the game presents us with a paradox.

In what follows, I will describe two solutions to the paradox. Both solutions involve modifying the values of the probabilities in the game, and so if a rational agent is defined as one who maximizes expected value (or expected utility) using the probabilities that are given, then these solutions won't count as rational procedures for determining how much to pay. Once the modifications are made, however, these solutions suggest that players are maximizing expected value. In any event, these procedures will direct a player to pay only a small amount to enter the game.

2 The terminal point

When a player pays some amount $w > 0$ to enter the game, the expected value of the game is not what is given in equation (1), but rather is the following.

$$\sum_{k=1}^{n} \left(\frac{1}{2}\right)^{k} \left(2^{k} - w\right) = \left(1 - \frac{w}{2}\right) + \left(1 - \frac{w}{4}\right) + \left(1 - \frac{w}{8}\right) + \ldots = \infty \quad (2)$$

Say that a player pays \$6. His or her payoff, if the game ends in one of the first nine rounds is in the middle row of table 1. The expected value of just that round is given in the next row, and the cumulative expected value up to that round is given in the final row.

A player who is deciding what to pay can be understood as selecting a *terminal point*: the round that has the lowest probability that the player believes or guesses might occur. The probabilities will, of course, affect the

round (n)	1	2	3	4	5	6	7	8	9
probability of n	50%	25%	12.5%	6.25%	3.13%	1.56%	0.781%	0.391%	0.195%
payoff (\$)	-4	-2	2	10	26	58	122	250	506
$E(n)$ (\$)	-2.00	-0.50	0.25	0.63	0.75	0.813	0.906	0.953	0.977
cumulative expected value (\$)	-2.00	-2.50	-2.25	-1.63	-0.81	0.09	1.047	2.023	3.012

Table 1: The payoffs for the first nine rounds if a player pays \$6 to enter the game.

choice of a terminal point. The possible payoffs can also affect the choice, but they might not (or their affect might be minimal). Once a terminal point is set, the probabilities for the rounds that come after the terminal point are treated as zero. So, in effect, the analysis of the game consists of only the rounds from the first to the terminal point. I will discuss the terminal point again, briefly, in the final section, but now we will turn to its application for selecting a payment to enter the St. Petersburg game.

3

First, we note that, rewritten, equation (2), is

$$\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{k}\left(2^{k}-w\right)=\left(\frac{1}{2^{n}}-1\right)w+n. \tag{3}$$

Next, once a player has set a terminal point, he or she has to decide how much to pay to enter the game. A player should not pay more than the expected value of the game up to his or her terminal point, n_t. So, setting the expected value equal to zero, we can solve for w^*.

$$E\left(\text{SPG}\right)=\left(\frac{1}{2^{n_t}}-1\right)w^*+n_t=0 \tag{4}$$

$$w^*=\frac{2^{n_t}\left(n_t\right)}{2^{n_t}-1} \tag{5}$$

For instance, if the player's terminal point is round 6, then $w^* = \$6.095$. If a player pays that amount, then the expected value of the game is zero, the same expected value as not playing. Paying less than w^* will produce a positive expected value for the game (up to round n_t). Assuming that not playing is preferred to playing if the expected value is zero for both options, the maximum amount that a player should pay to enter the game is slightly less than w^*. (This solution suggests, especially if a player's payment is made in increments of \$1, that the number of dollars that a player offers to pay will correspond to his or her terminal point.)

If the base unit (b) for the game is some amount other than \$2, then what a player is willing to pay to enter the game should change. For the same terminal point, larger values for b will mean (or at least allow for) larger payments to enter the game. But, compared to what it is in the standard version of the game—that is, when $b = \$2$—the player may change his or her terminal point since the game now has a different series of possible payoffs.

In any event, for this general version, the expected value of the game is

$$\sum_{k=1}^{n} \left(\frac{1}{2}\right)^k \left[(b)\left(2^{k-1}\right) - w\right] = \left(\frac{1}{2^n} - 1\right) w + \frac{bn}{2} \tag{6}$$

Setting the expected value equal to zero, we get

$$E\,(\text{SPG}) = \left(\frac{1}{2^{(n_t)}} - 1\right) w^* + \frac{bn_t}{2} = 0 \tag{7}$$

$$w^* = \frac{2^{(n_t-1)} b n_t}{2^{n_t} - 1} \tag{8}$$

So, for example, if $b = \$4$ and $n_t = 6$, then $w^* = \$12.1905$. When a player pays slightly less than w^*, say, \$12.19, then the expected value of the game—through six rounds—is \$0.0005. But if the player pays \$12.20, then the expected value of the game (for 6 rounds) is -\$0.0094.

round (n)	1	2	3	4	5	6	7	8	9
probability of n	50%	25%	12.5%	6.25%	3.13%	1.56%	0.781%	0.391%	0.195%
prize ($)	4	8	16	32	64	128	256	512	1,024
payoff ($)	-8	-4	4	20	52	116	244	500	1,012
$E(n)$ ($)	-4.00	-1.00	.50	1.25	1.63	1.81	1.91	1.95	1.98
cumulative expected value ($)	-4.00	-5.00	-4.50	-3.25	-1.63	0.19	2.09	4.05	6.02

Table 2: The payoffs through nine rounds if the base unit is $4 and a player pays $12 to enter the game.

4 A challenge to the expected utility solution

In the paper describing the St. Petersburg paradox, Daniel Bernoulli suggested that the paradox can be solved by using the utility that a player gets from each payoff rather than the payoff itself. Since, as Bernoulli explained, money has diminishing marginal utility, the expected *utility* of the game will have a limit. For instance, although it is not realistic, let's say that the utility for each payoff is set by log(*payoff*).[1] In this case, and ignoring w, the expected utility, as the game goes to infinity, is .602, which is the utility value for $4.[2] If, however, the base unit is changed to $\$10^{(2^n)}$—so the payoffs are $\$10^2$, $\$10^4$, $\$10^8$, $\$10^{16}$ etc.—the utility for each round will be 2, 4, 8, 16, etc., and the expected utility will go to infinity—in the same way that it does in equation (1) (Martin, 2017).

Setting aside the utility of the payoffs, and applying the terminal point solution to a game with $b = 10^{(2^n)}$ and the payoffs measured in dollars, we can make sense of how a player might approach such a game—although there

[1] The utility of a payoff should take into account the player's wealth, but log(*payoff*) will suffice for illustrating this version of the game.

[2]

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{2}\right)^k \left(\log 2^k\right) = \lim_{n\to\infty} \frac{\left(-n + 2^{n+1} - 2\right)(\log 2)}{2^n} = (2)(\log 2) = .602 \qquad (9)$$

round (n)	probability of n	payoff ($)	$E(n)$ ($)
1	50%	100	50
2	25%	10,000	2,500
3	12.5%	100,000,000	12,500,000
4	6.25%	10,000,000,000,000,000	$6.25 \cdot 10^{14}$
5	3.13%	100,000,000,000,000,000,000,000,000,000,000	$3.125 \cdot 10^{30}$
6	1.56%	$\$10^{64}$	$1.5625 \cdot 10^{62}$
7	0.78%	$\$10^{128}$	$7.8125 \cdot 10^{125}$
8	0.39%	$\$10^{256}$	$3.90625 \cdot 10^{253}$

Table 3: The payoffs for the first eight rounds of the St. Petersburg game when the base unit is $\$10^{(2^n)}$ and the payment is \$0.

are features of the game that make it less intuitive than the versions where the base unit is closer to \$2.

As in the other versions of the game, if the base unit is $\$10^{(2^n)}$, the expected value is infinite, but no one would be willing to pay any finite amount to enter the game.

$$\sum_{k=1}^{n} \left(\frac{1}{2}\right)^k \left[10^{(2^k)} - w\right] = \frac{1}{2}\left(10^{(2^1)} - w\right) + \frac{1}{4}\left(10^{(2^2)} - w\right) + \frac{1}{8}\left(10^{(2^3)} - w\right) + \ldots = \infty \quad (10)$$

So, this version of the game resembles the standard version in that sense. On the other hand, given the possible payoffs—a 25% chance at \$10,000, a 12.5% chance at \$100 million, a 6.25% chance at \$10 quadrillion, and so on—players may not think about this version of the game in the same way as they think about the standard version. Also, a player can change his or her terminal point based on the size of the payoff, and presumably, a player would do that if ever faced with this game.

In any case, the same solution described in section 3 can be used when $b = 10^{(2^n)}$. Setting the expected value equal to zero and solving for different values of n_t, we get the values for $w*$ listed in table 4.

terminal point (n_t)	**probability of** n_t	w^* (\$)
1	50%	100
2	25%	3,400
3	12.5%	14,288,628.57
4	6.25%	666,666,680,002,720
5	3.13%	$3.22581 \cdot 10^{30}$
6	1.56%	$1.5873 \cdot 10^{62}$

Table 4: For a game in which $b = 10^{(2^n)}$ and given a terminal point n_t, payments (w^*) that will make $E\,(\text{SPG}) = 0$.

As in the other versions of the game, given a terminal point, a player should pay no more than slightly less than w^*. In addition to the reasons mentioned above, this version of the game is also tricky to analyze because, given the size of the payoffs, what a player might want to pay and what a player would be capable of paying may conflict in a way that they don't in the standard version of the game.

5 A second solution

For the second solution, we forgo the terminal point and instead discount all of the probabilities using $prob' = (prob)^m$, $m > 1$. Thus, the expected value of the game is now this:

$$E\,(\text{SPG}) = \sum_{k=1}^{n} \left(\frac{1}{2^k}\right)^m \left(2^k - w\right) \tag{11}$$

So, if the probability is discounted with, say, $m = 1.3$, then, to have a positive expected value (for the entire game as it goes to infinity), a player should pay no more than \$6.32. (See table 5.)[3]

[3] Discounting probabilities for values less than 0.5 means that—to be consistent—the probabilities over 0.5 should be overvalued. For instance, if $prob'=(prob)^{1.25}$, then $\text{Pr} = 0.03125$ has an effective value of $\text{Pr} = 0.01314$. But if an event has a 3.125 percent chance of happening, then it has a 96.875 percent chance of not happening. In the same way, if the discounted probability is 1.314 percent, then the "discounted" probability of the event

m	w (\$)
1.2	8.72
1.3	6.32
1.4	5.12
1.5	4.41
1.6	3.93

Table 5: Values for m in $prob' = (prob)^m$. The values for w are the highest payments (given to the nearest cent) that will yield a positive expected value as the game goes to infinity. For instance, if $m = 1.3$ and $w = \$6.32$, the limit is \$0.00430809. If, however, $w = 6.33$, then the limit is $-\$0.0025305$.

Of course, this solution encounters a counterexample similar to the one used to challenge the expected utility solution. When $prob' = (prob)^m$, if the prizes are changed to $2^{(k^m)}$, then the expected payoff for each round is \$1 and the expected utility for the game goes to infinity.

These two solutions can also be combined. By calculating the expected value of the game up to a player's terminal point, we are, as stated earlier, treating the probability for the remaining rounds as zero. Alternatively, instead of zero, the probabilities for those rounds that follow the terminal point can be discounted with the function $prob' = (prob)^m$, $m > 1$. Then, the expected value for the whole game will approach a limit that is not much greater than the expected value of the game up to the terminal point.

$$E\,(\text{SPG}) = \sum_{k=1}^{n_t} \left(\frac{1}{2}\right)^k \left(2^k - w\right) + \sum_{k=n_t+1}^{n} \left(\frac{1}{2^k}\right)^2 \left(2^k - w\right) \tag{12}$$

For instance, if (a) a player's terminal point is 6, (b) the player pays \$6 to enter the game, and (c) the probabilities for the rounds after the terminal point are discounted with the function $prob' = (prob)^2$, then the expected value for the first six rounds is \$0.0938 and the expected value for the whole game as it goes to infinity is slightly less than \$0.109 . (I.e., the non-discounted probabilities are used for rounds $n = 1$ to $n = n_t = 6$, and the discounted

not happening should be 98.686 percent—so, it's value is increased. That may be correct, although if we are seeking procedures that actually describe how agents act, consistency is not a requirement.

probability, $prob' = (prob)^2$, is used for the rest of the series.)

$$E\,(\mathrm{SPG}) = \sum_{k=1}^{6} \left(\frac{1}{2}\right)^{k} \left(2^{k} - 6\right) + \sum_{k=n_t+1}^{n} \left(\frac{1}{2^{k}}\right)^{2} \left(2^{k} - 6\right) \approx 0.109 \qquad (13)$$

6

By using the terminal point, we are invoking an alleged feature of our psychology to explain the choices made by players in the St. Petersburg game. Typically, we expect reasonable agents to treat the probability of events that are very unlikely as zero (e.g., being in an airplane accident, being struck by lightning, or losing all of the value of one's investments in the stock market), and the terminal point solution is based on that tendency. Assuming that players are doing something like setting a terminal point, if we calculate the expected value up to a player's terminal point, then, within that restricted range, the choices that players make may very well follow the principle of maximizing expected value.

Nothing has been said yet about the terminal point that a player would actually select in the St. Petersburg game. Different players can select different terminal points, but, presumably, for most players, it will be one of the rounds with a probability between 3.13% and 0.195%. That corresponds to paying, at most, \$5 to \$9 (in the standard version of the game), which seems to fit with our intuition about what most people would be willing to pay.

One problem with the terminal point solution is that individuals who are thinking about the St. Petersburg game only volunteer what they would be willing to pay to enter. If the player's terminal point is simply inferred from the proposed payment, then, although it makes sense of the player's choice, we are not testing the claim that the player—deliberately or non-deliberately—set a terminal point and then selected a payment so that the game would have a positive expected value. Presumably, however, there are ways to separate setting the terminal point and selecting a payment so that the terminal point solution to the St. Petersburg paradox can be tested.

References

[1] Bernoulli, D. (1954) [1738]. Exposition of a New Theory on the Measurement of Risk. *Econometrica*, *22*, 23 – 36.

[2] Martin, R. (Fall 2017). The St. Petersburg Paradox. *The Stanford Encyclopedia of Philosophy*, Edward N. Zalta (ed.).