Probabilities and the St. Petersburg Paradox

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Start Here

Consider the following game, first described by Nicolaus Bernoulli. To participate, a player has to make a payment before the game begins. Then, a fair coin will be flipped until it lands showing heads. If the first flip is heads, then the player gets \$2 and the game ends. If it is tails, the player gets nothing and the game continues. If the coin is heads on the second flip, the player gets \$4 and the game ends. If it is tails, the player gets nothing and the game continues. And so on with the payoff doubling each round. This is the St. Petersburg game.

Please take a moment to answer these two questions.

(1) What is the maximum amount that you would be willing to pay to enter this game?

(2) Imagine that the game will be repeated 10 times. Guess the highest number of rounds that will occur in one of those games. Guess the lowest number of rounds that will occur in one of the games. (That is, the maximum and minimum number of times that the coin will be flipped. If for one of the games, the first flip is tails and the second is heads, then the game ends after two rounds.)

Write down your answers and put them in the envelope. Thanks!

Abstract

Two solutions to the St. Petersburg Paradox are described. First, when offered the chance to participate, players may be selecting a *terminal point*: the round in the game that has the lowest probability that a player believes or guesses might occur. Once that is set, the probabilities for the rounds after the player's terminal point are treated as 0. Alternatively, players can discount the probabilities for all possible rounds with the function $prob' = (prob)^m$, $m > 1$. After either modification to the player's evaluation of the probabilities, players may be following the principle of maximizing expected value when they offer only a small payment to enter the game.

Introduction

The expected value of the game is

$$\sum_{k=1}^{n}\left(\frac{1}{2}\right)^k\left(2^k\right)=\frac{1}{2}(2)+\frac{1}{4}(4)+\frac{1}{8}(8)+\ldots=1+1+1+\ldots=\infty. \quad (1)$$

Since the expected value is infinite, a rational player should be willing to pay any finite amount to play. Clearly, though, no one would pay a very large amount of money, and most people would be willing to pay very little. Hence, the game presents us with a paradox.

Beginning with Bernoulli's description of the declining marginal utility of money, solutions to this paradox have been given. I will describe two new solutions: the *terminal point solution* and the *discounted probabilities solution*.

The Terminal Point Solution

When a player pays some amount $w > 0$ to enter the game, the expected value of the game is the following.

$$\sum_{k=1}^{n}\left(\frac{1}{2}\right)^k\left(2^k-w\right)=\left(1-\frac{w}{2}\right)+\left(1-\frac{w}{4}\right)+\left(1-\frac{w}{8}\right)+\ldots=\infty \quad (2)$$

So, for example, if you pay \$6, and the game ends in the first round, your payoff is $-6+2=-\$4$.

If you pay \$6, and the game ends in the second round, your payoff is $-6+4=-\$2$.

If you pay \$6, and the game ends in the third round, your payoff is $-6+8=\$2$.

If you pay \$6, and the game ends in the fourth round, your payoff is $-6+16=\$10$. And so forth.

The Terminal Point

The first solution says that a player who is deciding what to pay is selecting a *terminal point*: the round that has the lowest probability that the player believes or guesses might occur. The probabilities will, of course, affect the choice of a terminal point. The possible payoffs can also affect the choice, but they might not (or their affect might be minimal). Once a terminal point is set, the probabilities for the rounds that come after the terminal point are treated as zero.

Once a player has set a terminal point, he or she has to decide how much to pay to enter the game. A player should not pay more than the expected value of the game up to his or her terminal point, n_t.

For instance, if the player's terminal point is round 6 (Pr = 1.56%), then the maximum that a player should pay is \$6.095. If a player pays that amount, then the expected value of the game is zero. Assuming that not playing is preferred to playing if the expected value is zero for both options, the maximum amount that a player with this terminal point should pay is slightly less than \$6.095.

tp	prob	max (\$)
3	12.5%	3.43
4	6.25%	4.27
5	3.13%	5.16
6	1.56%	6.09
7	0.78%	7.06

(This solution suggests, especially if a player's payment is made in increments of \$1, that the number of dollars that a player offers to pay is (or is close to) his or her terminal point.)

The Discounting Probabilities Solution

For the second solution, the game is understood to go to infinity (and so we longer have a terminal point). But all of the probabilities are discounted using $prob' = (prob)^m$, $m > 1$.

Thus, the expected value of the game is now this:

$$E\,(\text{SPG})=\sum_{k=1}^{n}\left(2^k-w\right)\left(\frac{1}{2^k}\right)^m \quad (3)$$

If the probability is discounted with, say, $m = 1.3$ and $w = 0$, then,

$$E\,(\text{SPG})=(2)\left(\frac{1}{2}\right)^{1.3}+(4)\left(\frac{1}{4}\right)^{1.3}+(8)\left(\frac{1}{8}\right)^{1.3}+$$

$$(16)\left(\frac{1}{16}\right)^{1.3}+\ldots=.81+.66+.54+.44+\ldots=\$4.33. \quad (4)$$

When $w > 0$, to have a positive expected value (for the entire game as it goes to infinity), a player should pay no more than \$6.32. (I.e., if $w = \$6.32$, $E\,(\text{SPG}) = 0$.)

The maximum amounts that a player should pay for different values of m are given in the table below.

m	max (\$)
1.2	8.72
1.3	6.32
1.4	5.12
1.5	4.41
1.6	3.93

Some Additional Math

Rewritten, equation (2), is

$$\sum_{k=1}^{n}\left(\frac{1}{2}\right)^k\left(2^k-w\right)=\left(\frac{1}{2^n}-1\right)w+n. \quad (5)$$

A player should not pay more than the expected value of the game up to his or her terminal point, n_t. So, setting the expected value equal to zero, we can solve for w^*.

$$E\,(\text{SPG})=\left(\frac{1}{2^{n_t}}-1\right)w^*+n_t=0 \quad (6)$$

$$w^*=\frac{2^{n_t}\,(n_t)}{2^{n_t}-1} \quad (7)$$

For the general version of the game (where the initial payoff, b, can be any amount), the expected value of the game is

$$\sum_{k=1}^{n}\left(\frac{1}{2}\right)^k\left[(b)\left(2^{k-1}\right)-w\right]=\left(\frac{1}{2^n}-1\right)w+\frac{bn}{2} \quad (8)$$

Setting the expected value equal to zero, we get

$$E\,(\text{SPG})=\left(\frac{1}{2^{(n_t)}}-1\right)w^*+\frac{bn_t}{2}=0 \quad (9)$$

$$w^*=\frac{2^{(n_t-1)}bn_t}{2^{n_t}-1} \quad (10)$$

So, for example, if $b = \$4$ and $n_t = 6$, then $w^* = \$12.1905$. When a player pays slightly less than w^*, say, \$12.19, then the expected value of the game—through six rounds—is \$0.0005. But if the player pays \$12.20, then the expected value of the game (for 6 rounds) is -\$0.0094.

References

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